The sphinx: a limit-periodic tiling of the plane

1989 J. Phys. A: Math. Gen. 22 L1163

(http://iopscience.iop.org/0305-4470/22/24/006)

J. Phys. A: Math. Gen. **22** (1989) L1163–L1166. Printed in the UK

LETTER TO THE EDITOR

The sphinx: a limit-periodic tiling of the plane

C Godrèche

Service de Physique du Solide et de Résonance Magnétique, Commissariat à l'Energie Atomique, Saclay, F-91191 Gif-sur-Yvette Cedex, France

Received 16 October 1989

Abstract. The sphinx is a non-periodic tiling of the plane made of one type of tile. In order to characterise the type of order found in this tiling, the computation of its diffraction spectrum is considered. The spectrum contains a discrete component, with Bragg peaks located at the vertices of triangular lattices with lattice spacings, in reciprocal space, equal to any power of $\frac{1}{2}$.

A question which naturally arises when looking at aperiodic tilings of the plane is the nature of their diffraction spectra. For tilings generated by a projection method, the answer to this question is easy. In contrast, this question may be a challenging one for tilings for which the only known construction uses an inflation method.

We consider such a case here. The tiling of figure 1 may be found in an article by Gardner [1], who named it the 'sphinx', or in the book by Grünbaum and Shephard [2]. It is generated by inflation, and made of a single type of tile, which tiles the plane non-periodically. The proof of this is given in [2]. In contradistinction to the Penrose tiling, this single tile may also generate a periodic tiling: two tiles abutted upside down form a parallelogram.

The method used here to compute the Fourier transform is an extension to two dimensions of a method given by Bombieri and Taylor [3] to answer the following question. Consider a one-dimensional structure generated by inflation, i.e. by associating short and long bonds with a substitution acting on letters. Then, if one knows that there is a discrete part (Dirac peaks) in the Fourier spectrum of the structure, what constraint does that imply for the substitution? This method [4] has already been used to compute the Fourier transform of tilings made with Robinson tiles: the Penrose

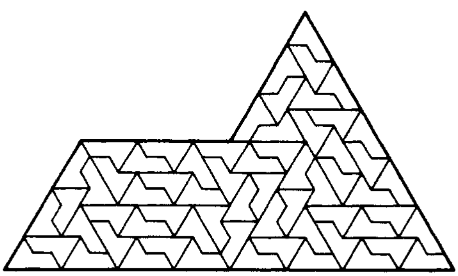

Figure 1. The Sphinx tiling.

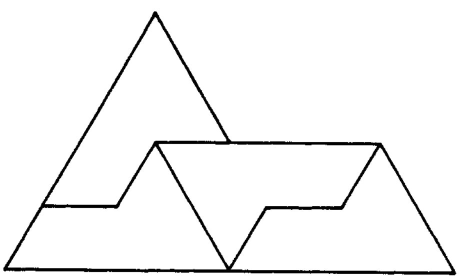

Figure 2. Composing four sphinxes at generation n gives a sphinx at generation $n+1$ with twice linear dimensions.

tiling, and also more general tilings such as chiral and random tilings. Five- or seven-fold tilings generated by inflation, with non-discrete Fourier specta [5] have also been investigated recently.

The tiling looked at here is much simpler, and for this reason it has the virtue of illustrating the method in a simple and pedagogical manner.

Throughout this paper, we treat as synonymous the words inflation, substitution and composition. The sphinx tiling is obtained by a simple rule: four sphinxes put together give, by composition, one bigger sphinx (see figure 2). One has, though, to distinguish between a right and a left sphinx, depending on the position of the 'head' of the sphinx. So, a right sphinx is made out of three left sphinxes and a right sphinx; the converse is true for a left sphinx. This may be described by a substitution acting on the tiles at generation n (i.e. after n iteration steps), A_n and B_n, representing the right and left sphinx respectively. Hence the tiles at generation $n+1$ are obtained from those at generation n by

$$A_{n+1}=A_n+3B_n \qquad B_{n+1}=3A_n+B_n \tag{1}$$

using an obvious notation. These equations only take into account the numbers of tiles involved in the substitution. They will be completed below by a more complete geometrical description. The substitution matrix

$$M=\begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix} \tag{2}$$

gives the numbers of tiles of generation n found in the tile of generation $n+1$. The eigenvalues of M are 4 and -2.

By convention, let every tile have its origin at the left corner; $\hat{e}_k$ ($k=0,\ldots,5$) are the unit vectors of the regular hexagonal lattice; R^m ($m=0,\ldots,5$), denotes rotation through angle $m\pi/3$ in the plane. In its initial position, a tile has its largest side horizontal. The geometrical description of one step of the composition is expressed by the equations

$$\begin{aligned}A_{n+1}&=[R^2,(3\hat{e}_0+\tfrac{1}{2}\hat{e}_2)2^n]A_n+[R^0,\mathbf{0}]B_n+[R^0,\tfrac{3}{2}\hat{e}_0 2^n]B_n+[R^3,(\tfrac{3}{2}\hat{e}_0+\hat{e}_1)2^n]B_n\\ B_{n+1}&=[R^0,\mathbf{0}]A_n+[R^0,\tfrac{3}{2}\hat{e}_0 2^n]A_n+[R^3,(3\hat{e}_0+\hat{e}_2)2^n]A_n+[R^4,2\hat{e}_1 2^n]B_n\end{aligned} \tag{3}$$

which give the content of the tile at generation $n+1$ in terms of tiles at generation n. In the brackets, the first terms are rotations, and the second ones translations, acting on a tile at generation n, to bring it to its new position in the tile of the next generation. Here, A_n and B_n are condensed notation for $A_n(\boldsymbol{x})$ and $B_n(\boldsymbol{x})$ and represent the densities of matter attached to the right and left tiles respectively. For instance, the densities may be peaked distributions on the barcentre of each tile. In other terms, the tiling we consider is the dual of the given tiling.

Since the linear dimensions of the inflated tile is doubled at each step of inflation, the linear dimensions of a tile at generation n is proportional to 2^n.

The Fourier transform of the densities A_n and B_n are the Fourier amplitudes of the tiles denoted $\tilde{A}_n(\boldsymbol{q})$ and $\tilde{B}_n(\boldsymbol{q})$. Then, from (3), one gets the recurrence relations between the Fourier amplitudes at successive generations:

$$\begin{aligned}\tilde{A}_{n+1}(\boldsymbol{q})&=\exp(-\mathrm{i}\boldsymbol{q}\cdot(3\hat{e}_0+\tfrac{1}{2}\hat{e}_2)2^n)\tilde{A}_n(R^4\boldsymbol{q})+\tilde{B}_n(\boldsymbol{q})\\ &\quad+\exp(-\mathrm{i}\boldsymbol{q}\cdot\tfrac{3}{2}\hat{e}_0 2^n)\tilde{B}_n(\boldsymbol{q})+\exp(-\mathrm{i}\boldsymbol{q}\cdot(\tfrac{3}{2}\hat{e}_0+\hat{e}_1)2^n)\tilde{B}_n(R^3\boldsymbol{q})\\ \tilde{B}_{n+1}(\boldsymbol{q})&=\tilde{A}_n(\boldsymbol{q})+\exp(-\mathrm{i}\boldsymbol{q}\cdot\tfrac{3}{2}\hat{e}_0 2^n)\tilde{A}_n(\boldsymbol{q})\\ &\quad+\exp(-\mathrm{i}\boldsymbol{q}\cdot(3\hat{e}_0+\hat{e}_2)2^n)\tilde{A}_n(R^3\boldsymbol{q})+\exp(-\mathrm{i}\boldsymbol{q}\cdot 2\hat{e}_1 2^n)\tilde{B}_n(R^2\boldsymbol{q}).\end{aligned} \tag{4}$$

We investigate now whether the Fourier spectrum contains a discrete part. If so, there exist values of $\boldsymbol{q}$ for which the Fourier amplitudes have maximum growth, i.e. proportional to the number of tiles in the sample at a given generation n, namely 4^n. This implies that all the phases in (4) must vanish, mod 2π, for these particular values of $\boldsymbol{q}$. The proof of this assertion, analogous to that used in the one-dimensional case, is left to the reader. These conditions yield the following equations on $\boldsymbol{q}$:

$$\begin{aligned}
\boldsymbol{q}\cdot(3\hat{e}_0+\tfrac{1}{2}\hat{e}_2)2^n &\to 0 \bmod 2\pi\\
\boldsymbol{q}\cdot\tfrac{3}{2}\hat{e}_0 2^n &\to 0 \bmod 2\pi\\
\boldsymbol{q}\cdot(\tfrac{3}{2}\hat{e}_0+\hat{e}_1)2^n &\to 0 \bmod 2\pi\\
\boldsymbol{q}\cdot(3\hat{e}_0+\hat{e}_2)2^n &\to 0 \bmod 2\pi\\
\boldsymbol{q}\cdot 2\hat{e}_1 2^n &\to 0 \bmod 2\pi.
\end{aligned} \tag{5}$$

In order to solve these equations, we use the fact that the values of x such that

$$x2^n \to 0 \bmod 1 \qquad \text{when } n\to\infty \tag{6}$$

have the form

$$x=\frac{M}{2^N} \tag{7}$$

for some rank N, and some odd integer M; i.e. x is a dyadic. Note that for $n \geqslant N$, equation (6) becomes an equality. The solution to (5) is easily found to be

$$\frac{\boldsymbol{q}}{2\pi}=\frac{1}{2^\nu}(\lambda\hat{e}_0+\mu\hat{e}_1) \tag{8}$$

where ν is some integer, and

$$\lambda=4L \qquad \mu=4(M-2L) \tag{9}$$

with L and M being two integers. These results show that there exists a discrete part in the spectrum, for the values of $\boldsymbol{q}$ given by (8). This spectrum is therefore almost periodic. In fact, since the frequencies appearing in (8) are powers of $\frac{1}{2}$, we encounter here a particular case of an almost periodic spectrum, called limit periodic [6].

The end of the calculation is performed along the same lines as in [4], i.e. for $\boldsymbol{q}$ given by (8), with a given ν, when $n \geqslant \nu$ equations (4) involve a constant 12×12 matrix with integer entries. Indeed, there are two equations, which connect $\boldsymbol{q}$ to $R_{\boldsymbol{q}}$, $R^2\boldsymbol{q},\ldots,R^5\boldsymbol{q}$. This matrix may be diagonalised by means of the superpositions

$$\tilde{A}_{n,p}(\boldsymbol{q})=\sum_{k=0}^{5}\omega^{pk}\tilde{A}_n(R^k\boldsymbol{q}) \tag{10}$$

where $\omega=\exp(-\mathrm{i}\pi/3)$, and an analogous expression for the left amplitudes. The constant 12×12 matrix is decomposed into six 2×2 blocks

$$\begin{pmatrix}\tilde{A}_{n+1,p}\\ \tilde{B}_{n+1,p}\end{pmatrix}=\begin{pmatrix}\omega^{2p} & 2+\omega^{3p}\\ 2+\omega^{3p} & \omega^{4p}\end{pmatrix}\begin{pmatrix}\tilde{A}_{n,p}\\ \tilde{B}_{n,p}\end{pmatrix}. \tag{11}$$

The 2×2 matrix appearing in (11) has the following eigenvalues:

$$\begin{aligned}
&\lambda_{0,i}=4,\ -2 \qquad \lambda_{1,i}=\lambda_{5,i}=0,\ -1\\
&\lambda_{2,i}=\lambda_{4,i}=\tfrac{1}{2}(-1\pm\sqrt{33})=2.372,\ -3.372 \qquad \lambda_{3,i}=2,\ 0
\end{aligned} \tag{12}$$

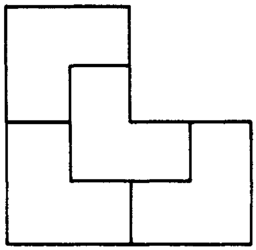

Figure 3. Composition of four identical tiles leading to a non-periodic tiling.

where $i=1, 2$. The original Fourier amplitudes are recovered by inversion of (10)

$$\tilde{A}_n(R^k\boldsymbol{q}) = \frac{1}{6}\sum_{p=0}^{5} \omega^{-pk}\tilde{A}_{n,p}(\boldsymbol{q}) \qquad 0 \leqslant p \leqslant 5. \tag{13}$$

The $p=0$ term grows as 4^n and yields a delta peak with a k-independent amplitude. Thus the Fourier transform has hexagonal symmetry.

Since the dilatation factor is integer, it is in principle possible to compute the Fourier amplitudes explicitly. Indeed, we already remarked that as soon as $n \geqslant \nu$ the phases in (5) vanish mod 2π identically.

One should of course make the same caveat as for the one-dimensional case studied by Bombieri and Taylor, i.e. this argument does not rule out the presence of a continuous part in the spectrum.

Another tiling [2] similar to the sphinx but with square symmetry may be obtained by the construction shown in figure 3. Note that now the tiling is made with only *one* type of tile, even in a restricted sense, namely that one does not have to distinguish between a right and a left tile. The computation of the Fourier spectrum is thus even simpler than for the sphinx. One finds a limit periodic spectrum with square symmetry as expected.

It is worth mentioning that the 1D structure built upon the substitution given in (1) has a different Fourier spectrum than that of the sphinx. Indeed, we remark that the second eigenvalue, -2, does not play any role in the 2D case, whereas in 1D this eigenvalue appears in the conditions of vanishing phases. Though the largest eigenvalue, 4, is an algebraic number or order one, hence a Pisot number [7], the presence of a second eigenvalue with modulus greater than 1 mimics the non-Pisot case.

It is a pleasure to thank J M Luck for helpful remarks on this work, M Senechal for her encouragement to write it, and J Taylor for a critical reading of the manuscript.

References

[1] Gardner M 1977 *Sci. Am.* **236** 110

[2] Grünbaum B and Shephard G C 1987 *Tilings and Patterns* (San Francisco: Freeman) ch 10

[3] Bombieri E and Taylor J E 1986 *J. Physique Coll.* **C3** 19; 1987 *Contemp. Mat.* **64** 241

[4] Godrèche C and Luck J M 1989 *J. Stat. Phys.* **55** 1

[5] Godrèche C and Luck J M 1989 *Proc. Anniversary Adriatico Research Conf. on Quasicrystals, Trieste, July 4–7*

[6] Levitan B M and Zhikov V V 1982 *Almost Periodic Functions and Differential Equations* (Cambridge: Cambridge University Press)

[7] Cassels J W S 1957 *An Introduction to Diophantine Approximation* (Cambridge: Cambridge University Press)